CHARACTERISTICS OF BROKEN-LINE APPROXIMATION AND ITS USE IN DISTORTION AUDIO EFFECTS

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ABSTRACT

This paper deals with an analytic solution of spectrum changes in scalar non-linear discrete systems without memory, whose transfer characteristics can be approximated via broken-line function. The paper also deals with relations between the harmonics ratio and the approximation parameters. Furthermore, the dependence of the harmonics ratio on the amplitude of a harmonic input signal is presented for the most common characteristics that are approximated via broken-line function. These characteristics are judged from the dissonance point of view.

1. INTRODUCTION

We have to deal with aliasing when a digital signal is being processed by a non-linear system. The aliasing is caused by bandwidth extension when the highest frequency component exceeds half the sampling frequency. To prevent that, we can either upsample the input signal or approximate the transfer function of the system via the finite sum of terms of Taylor's series and use nonlinear processing by band-limiting input signal range as published in [1].

That is why non-linear systems are used with such a type of approximation whose response to an input signal with limited bandwidth has a limited bandwidth as well (e.g. polynomial approximation) or with such a type of approximation which ensures that harmonics of a higher order than n are masked by harmonics of a lower order than n or with non-stationary spectrum components (e.g. exponential approximation [2]).

The polynomial and exponential approximations have the advantage that approximation parameters can be evaluated by solving a linear equation system according to the required ratio of harmonics [2]. On the contrary, the ratio cannot be set-up independently for each harmonic when broken-line approximation is used. Furthermore, the response of such a system to a limited-bandwidth input signal has not a limited bandwidth (see below). However, the computing-power demands of these systems are low. An analytic solution to the computation of amplitude of higher harmonics will be presented below as well as the common spectrum types of output signal, which can be produced by a non-linear system with broken-line approximation as a response to the harmonic input signal.

2. BROKEN-LINE APPROXIMATION

Broken-line approximation of a non-linear transfer function $\Psi()$ is defined using R linear sections, for which the following equation holds for $i = 1, 2, ..., R$

$$y[n] = S_i\left(x[n] - x_{\mathrm{P}i}\right) \ \text{ for } x_{i-1} \le x[n] < x_i \,, \tag{1}$$

where S_i are the slopes of straight lines in the area $x_{i-1} \le x[n] < x_i$, $x_{\mathrm{P}i}$ are the points in which given line cuts the x axis, and x_i are the lower limits of a particular section of the function (see Figure 1). In the discrete domain, the spectrum changes can be evaluated in such systems by computing the approximation of coefficients of the discrete Fourier transform [3].
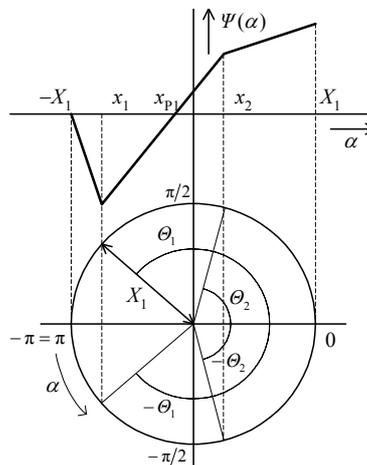

Figure 1: *Drawing up the output signal equation of non-linear system with transfer characteristics approximated by broken-line function using limit angles.*

The even-function attribute of the Fourier series [3] can be utilized in the case of spectral component analysis of the output signal of a non-linear system with cosine input signal (which is an even function). It can be seen from Figure 1 that the period of input signal $x(\alpha)$ is 2π and the function $\Psi(x)$ is identical for $x(\alpha)$ and for $x(-\alpha)$. The following equation holds for the approximation of the discrete Fourier transform coefficients of output signal

$$Y(\alpha) = \frac{1}{\pi} \int_{-\pi}^{\pi} y(\alpha)\cos k\alpha\, \mathrm{d}\alpha = \tag{2}$$
$$= \frac{1}{\pi} \int_{-\pi}^{0} y(\alpha)\cos k\alpha\, \mathrm{d}\alpha + \frac{1}{\pi} \int_{0}^{\pi} y(\alpha)\cos k\alpha\, \mathrm{d}\alpha$$

If we substitute equation (1) to equation (2) we obtain the following equation

$$Y(\alpha) = \frac{X_1}{\pi} \sum_{i=0}^{R-1} S_i \left(\int_{-\Theta_i}^{-\Theta_{i+1}} \mathrm{P}_{k,i}(\alpha)\mathrm{d}\alpha + \int_{\Theta_{i+1}}^{\Theta_i} \mathrm{P}_{k,i}(\alpha)\mathrm{d}\alpha \right), \tag{3}$$

where $\Theta_0 = \pi$, $\Theta_R = 0$, and

$$P_{k,i}(\alpha) = \cos\alpha \cos k\alpha - \frac{x_{Pi}}{X_1}\cos k\alpha. \qquad (4)$$

Using several goniometrical identities we obtain the following equation for the amplitude of k-th harmonics for $k > 1$ (see [2] for details)

$$Y_k(\alpha) = \frac{2X_1}{\pi}\sum_{i=1}^{R-1} \frac{S_i - S_{i-1}}{k^2-1}\big(k\sin k\Theta_i \cos\Theta_i - \sin\Theta_i \cos k\Theta_i\big) - \\ -\frac{S_i x_{Pi} - S_{i-1}x_{Pi-1}}{kX_1}\sin k\Theta_i, \qquad (5)$$

where $\alpha = 2\pi n/N$, N is the length of the processed signal, X_1 is the amplitude of the harmonic input signal, and $\cos\Theta_i = x_i/X_1$. The following equations hold for the amplitude of the first harmonic and the dc component [2]

$$Y_1(\alpha) = \frac{X_1}{\pi}\sum_{i=1}^{R-1} (S_i - S_{i-1})(\Theta_i + \cos\Theta_i \sin\Theta_i) - \\ -\frac{2}{X_1}(S_i x_{Pi} - S_{i-1}x_{Pi-1})\sin\Theta_i \qquad (6)$$

$$\frac{Y_0(\alpha)}{2} = \frac{X_1}{\pi}\sum_{i=1}^{R-1}(S_i - S_{i-1})\sin\Theta_i - (S_i x_{Pi} - S_{i-1}x_{Pi-1})\frac{\Theta_i}{X_1}. \qquad (7)$$

2.1. Characteristics and Parameters of Approximation

It can be seen from equation (5) that the output signal of a non-linear system with transfer characteristics approximated by broken-line function has not a limited bandwidth if $R > 1$. Equation (5) is a sum of goniometric functions, and a period ξ of spectral component amplitude repetition can be found for a finite number of limit angles Θ_i. However, their amplitudes decrease very slowly. The highest harmonic of the output signal spectrum that is not masked by lower harmonics can be found using the psycho-acoustical model. The upsampling ratio can be chosen according to the order of this harmonic.

Furthermore, equation (5) shows that the amplitudes of harmonics depend on the difference of slopes $S_i - S_{i-1}$ of adjacent linear sections, rather than on the difference $S_i x_{Pi} - S_{i-1}x_{Pi-1}$. So the amplitudes of higher harmonics increase with the difference of right and left limits at the points of function discreteness. The following equations hold for $i = 0,1,..., R-1$ if the approximation function is continuous

$$S_{i-1}(x_i - x_{Pi-1}) = S_i(x_i - x_{Pi}), \qquad (8)$$

i.e. the S_i and x_{Pi} parameters are linearly dependent according to the equation

$$x_{Pi} = x_i + \frac{S_{i-1}}{S_i}(x_{Pi-1} - x_i). \qquad (9)$$

By equation (5) one could say that the amplitudes of output signal harmonics are linearly dependent. However, the substitutions $x_0 = -X_1$ a $x_R = X_1$ were used when equation (5) was derived (see Figure 1). So the input signal amplitude changes influence all limit angles Θ_i. The input signal will not span the i-th section of the characteristic if $|\cos\Theta_i| > 1$. The step-change of spectrum character of the output signal can be seen from the joint amplitude-frequency analysis in Figure 2, when the input signal amplitude exceeds the limit level $\cos\Theta_i$. The properties of the odd and the even transfer characteristic function can be also demonstrated using equation (5). With the harmonic input signal, only odd

harmonics will be in the output signal when $\Psi(\alpha) = -\Psi(-\alpha)$, and only even harmonics will be in the output signal when $\Psi(\alpha) = \Psi(-\alpha)$ (see [2] for proof).
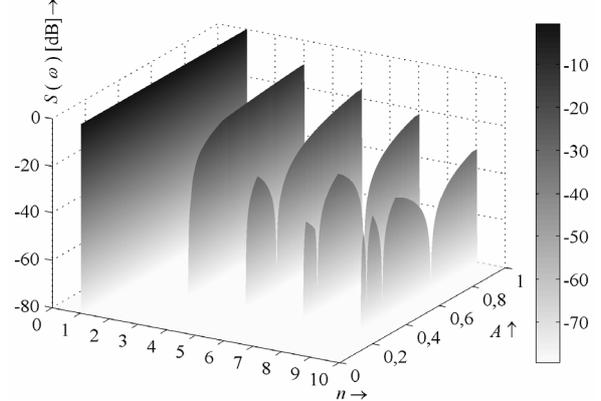

Figure 2: *Joint amplitude-frequency analysis of output signal of symmetrical limiter when* $\cos\Theta = 0.3$.

In the common case the broken-line approximation of transfer characteristic with R linear sections has $3R-1$ parameters. The number of parameters decreases when the S_i and x_{Pi} parameters are linearly dependent according to equation (8). The Following equations hold for the x_{P0} a S_{R-1} parameters if the output signal range is $\langle y_{min}; y_{max}\rangle$

$$x_{P0} = -X_1 - y_{min}/S_0 \\ S_{R-1} = y_{max}/(X_1 - x_{PR-1}) \qquad (10)$$

The total number of parameters of broken-line continuous function with R sections is $2R-2$.

2.2. Characteristics of Typical Broken-Line Approximations

Figure 3 shows the transfer characteristics of a simple non-linear system.
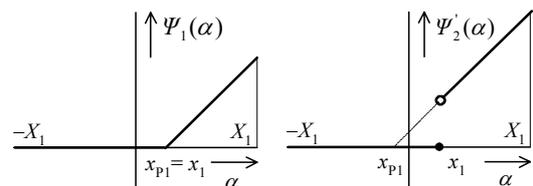

Figure 3: *Transfer characteristics of simple non-linear system.*

Equations for computing the amplitudes of output signal harmonics for such a type of system can be found in various publications dealing with analogue non-linear systems (e.g. in [4]). We can obtain the same equations for $\Psi_1(\alpha)$ from equation (5)

$$\frac{Y_k(\alpha)}{Y_1(\alpha)} = \frac{2}{k(k^2-1)}\frac{\sin k\Theta \cos\Theta - k\sin\Theta \cos k\Theta}{\Theta - \sin\Theta \cos\Theta}. \qquad (11)$$

The following equation holds for a modified function $\Psi_1'(\alpha)$

$$Y_k'(\alpha) = Y_k(\alpha) + \frac{2S\delta}{k\pi}\sin k\Theta \text{ for } k = 1,2,..., \qquad (12)$$

where $\delta = x_1 - x_{P1}$. Figure 4 shows the joint amplitude-frequency analysis of output signal of a non-linear system with this type of transfer characteristic. The level of the output signal is zero for

input signal amplitudes below x_1. The amplitudes of higher harmonics are high if level x_1 is slightly exceeded, and decrease with increasing harmonic input signal amplitude.
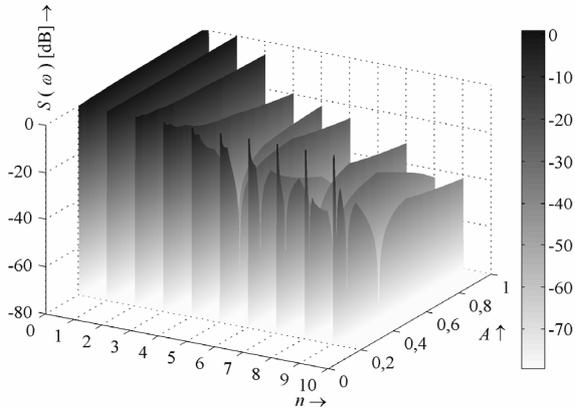

Figure 4: *Joint amplitude-frequency analysis of output signal of non-linear system from Figure 3.*

Figure 5 shows an interesting output signal spectrum that we obtain for $x_1 = 0$. In this case, the spectrum of the output signal consists of the first and the even harmonics only and their amplitude ratio does not depend on the amplitude of the harmonic input signal.
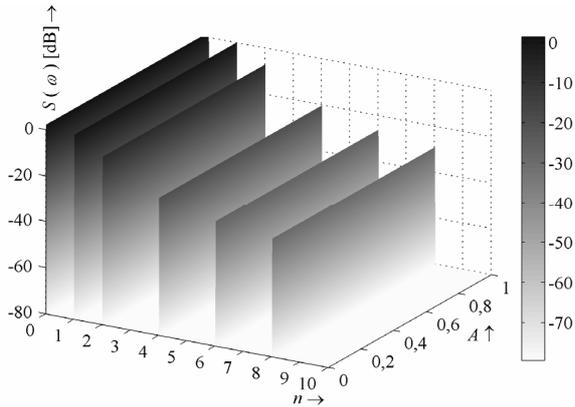

Figure 5: *Joint amplitude-frequency analysis of output signal of non-linear system from Figure 3 with $x_1 = x_{P1} = 0$.*

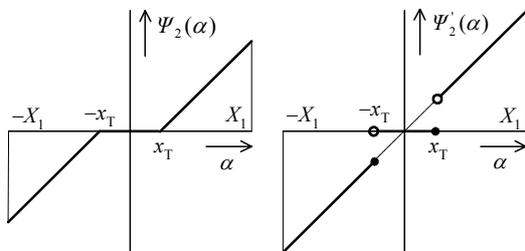Figure 6 shows the transfer characteristics of a system with soft and hard thresholds.

Figure 6: *Transfer characteristics of system with soft and hard threshold.*

We can derive equations for this system from equation (5), which are similar to equations (11) and (12). However, the even harmonics have zero amplitude and the amplitude of the odd harmonics is doubled.

On the contrary, the amplitudes of the odd harmonics are zero and the amplitudes of the even harmonics are doubled in comparison with equations (11) and (12) when the transfer characteristics are abs($\Psi_2(\alpha)$) and abs($\Psi_2'(\alpha)$) (see [2] for proof).

Typical transfer characteristics of non-linear system commonly used for distortion audio effects are in Figure 7. We can derive relatively complicated equations for amplitudes of the harmonics of output signal spectrum of a non-symmetrical limiter ($x_{T1} \neq -x_{T2}$) from equation (5) (see [2] for details).
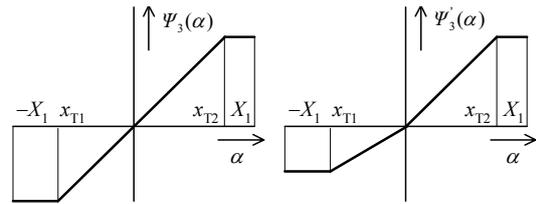

Figure 7: *Transfer characteristics of non-symmetrical limiter and limiter with non-linearity around operating point.*

Figure 8 shows the joint amplitude-frequency analysis of the output signal of non-symmetrical limiter.
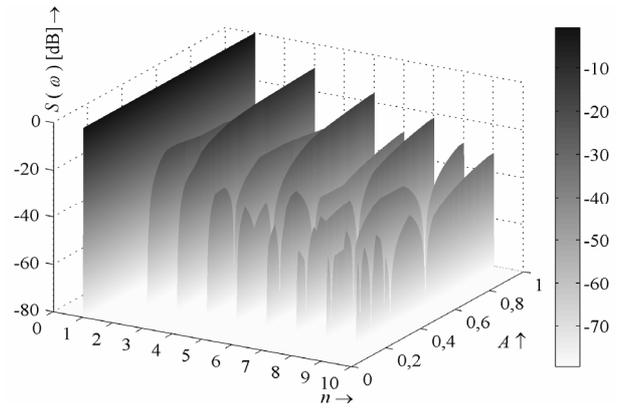

Figure 8: *Joint amplitude-frequency analysis of output signal of non-symmetrical limiter.*

Simpler equations can be derived for the symmetrical limiter with transfer function $\Psi_3(\alpha)$ when $x_{T1} = -x_{T2}$

$$Y_k(\alpha) = \frac{2S_1X_1}{\pi(k^2-1)}\left(1+(-1)^{k+1}\right)\left(k\sin k\Theta_2 \cos\Theta_2 - \sin\Theta_2 \cos k\Theta_2\right) \quad (13)$$

$$Y_1(\alpha) = \frac{2S_1X_1}{\pi}\left(\Theta_2 + \sin\Theta_2 \cos\Theta_2\right) \quad \frac{Y_0(\alpha)}{2} = 0 \cdot \quad (14)$$

One can see from equations (13) that the amplitudes of the even harmonics are zero (see Figure 2). On the contrary, the amplitudes of the odd harmonics of symmetrical limiter with transfer function abs($\Psi_3(\alpha)$) when $x_{T1} = -x_{T2}$ are zero. It can be seen from the following equations derived for such a type of system from equation (5)

$$Y_k(\alpha) = \frac{2S_1X_1}{\pi(k^2-1)}\left(1+(-1)^k\right)\left(k\sin k\Theta_2 \cos\Theta_2 - \sin\Theta_2 \cos k\Theta_2\right) \quad (15)$$

$$Y_1(\alpha) = 0 \quad \frac{Y_0(\alpha)}{2} = \frac{2S_1 X_1}{\pi} \left(\Theta_2 \cos \Theta_2 + \sin \Theta_2 \right) \cdot \qquad (16)$$

The transfer characteristics of all systems mentioned above have a linear section around the operating point. That is why only the first harmonic (or no signal) is present at the system output until the amplitude of the input harmonic signal exceeds the first limit point. Figure 9 shows the joint amplitude-frequency analysis of output signal of limiter with transfer characteristics $\Psi'_3(\alpha)$ from Figure 7, which has non-linearity around the operating point.

It can be seen that the output signal spectrum for the amplitudes of harmonic input signal below the first limit point is similar to the output signal spectrum of the non-linear system from Figure 3 with $x_1 = x_{P1} = 0$. Higher harmonics ratio depends on the difference of slopes $S_i - S_{i-1}$ as mentioned in section 2.1.

2.3. Distortion DAFx Using Broken-Line Approximation

The scalar non-linear discrete system without memory, whose transfer characteristic can be approximated via broken-line function, can be used in any nonlinear audio processor.

The question is which type of approximation should be used for the distortion audio effect. Several typical approximations used in these effects are described in [5] which start from analogue prototypes. However, we can design a non-linear system that generates higher harmonics according to our requirements using equation (5) and equations derived from it. We assume that such a type of spectrum enhancement of the output signal is required that is not perceived unpleasantly. Furthermore, we assume that such an upsampling ratio is used that the aliasing spectrum components are masked by the harmonic components.

There are several criteria for the valuation of non-linear distortion of a system, e.g. simple valuation using weighted harmonic distortion. The valuation using the dissonance ratio is another type of valuation. It is most frequently determined as the multiplication of numerator and denominator of a fraction that determines the interval between two pure tones. A 2D histogram can be obtained if this valuation is applied to intervals between the harmonic spectrum components. The dissonance ratio increases with the number of the harmonic, it is lower with the even harmonics, and it is highest with the seventh harmonic (see [2] for details).

According to this valuation, the system should mainly generate the even harmonics but the odd harmonics should not be suppressed. It follows from text above and from [5] as well that this can be achieved using the system with non-symmetrical signal limiting from Figure 7. A faster decrease in the amplitude of higher harmonics than with a classical limiter can according to (8) be achieved by increasing the slope of the transfer characteristics section, which performs the signal limiting. The generation of higher harmonics even for low amplitudes of the input signal can be achieved using a non-linear system with non-linearity around the operating point (see Figure 9).

Following equation describes the non-linear system designed for the distortion effects. The equation was designed according to characteristics of the broken-line approximation described above:

$$y[n] = \begin{cases} x[n](1-d_2) + x_{\mathrm{T}}(d_1 - d_2) & \text{for } x[n] \le -x_{\mathrm{T}} \\ x[n](1-d_1) & \text{for } -x_{\mathrm{T}} < x[n] \le 0 \\ x[n] & \text{for } 0 < x[n] \le x_{\mathrm{T}} \\ x[n](1-d_2) + x_{\mathrm{T}} d_2 & \text{for } x_{\mathrm{T}} < x[n], \end{cases} \qquad (17)$$

where $d_1, d_2 \in (0,1)$, x_{T} is the threshold level, d_1 is distortion ratio bellow this level and d_2 is the ratio above the threshold level.

In contrast to the common symmetrical limiters, the output signal of such system consists of the first and even harmonics only when amplitude of the harmonic input signal is bellow x_{T}. The distortion ratios above and bellow the threshold level can be adjusted almost independently. Higher harmonics with high dissonance ratio are attenuated when $d_1 < 0.1$.
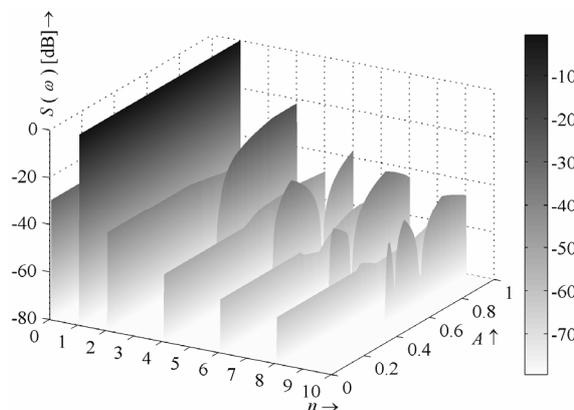

Figure 9: *Joint amplitude-frequency analysis of output signal of limiter with non-linearity around operating point.*

3. CONCLUSION

A direct realization of discrete non-linear systems with transfer characteristics approximated by broken-line function is controversial because of the aliasing distortion caused by the unlimited bandwidth of output signal of the system. However, we can easily modify the ratio of amplitudes of the output signal harmonics via simple changes of the approximation parameters while keeping the changes of the type of output signal spectrum under our control. The aliasing distortion can be suppressed using the input signal upsampling, whose ratio is determined using the psychoacoustical model, or we could find relations between the broken-line approximation parameters and the coefficients of its Taylor series. Future work will be focused on examining the output signal spectrum changes when transfer characteristic smoothing is used.

4. ACKNOWLEDGEMENTS

5. REFERENCES

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