DIGITAL WAVEGUIDE NETWORKS AS MULTIDIMENSIONAL WAVE DIGITAL FILTERS

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1. INTRODUCTORY REMARKS

Multidimensional wave digital filters (MDWDFs) [1, 2] have recently been applied toward the numerical simulation of distributed systems. The basic procedure for deriving an algorithm is similar to that which was originally developed for deriving wave digital filter [3, 4] structures from lumped analog networks, though in that case, the application was to filter design, and not explicitly to simulation. In the lumped case, one begins from a given analog network structure, composed typically of RLC elements (and possibly more exotic devices such as transformers, gyrators, etc.). One then appl;ies a continuous-to-discrete spectral mapping to each reactive element, and, after the introduction of *wave variables* [3], ends up with a recursible filter structure. The spectral mapping (a particular type of bilinear transform, which corresponds, in the discrete-time domain, to the use of the trapezoid rule of numerical integration) is chosen so that the energetic properties of the analog network are mirrored by the discrete-time structure. It has the form

$$s \rightarrow \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} \tag{1}$$

where s is the continuous-time frequency variable and z^{-1} is the unit delay in the frequency domain. T is the sampling period. Indeed, the digital filter topology is the same as that of the original analog filter, and structures so derived possess many desirable numerical properties (including guaranteed stability even in a finite word-length computer implementation) [4]. In effect, such a filter is an explicit numerical integration scheme for the system of ordinary differential equations (ODEs) which describes the time evolution of the state of the analog network.

The procedure is very similar in multiple dimensions (MD). For simulation purposes, however, a suitable multidimensional Kirchoff circuit (MDKC) representation [2] of a given system of partial differential equations (PDEs) must be first found; discretization is then performed in the same way as in the lumped case, namely through generalized MD spectral transformations which preserve passivity in a MD sense [5], and the introduction of wave variables. A bare-bones summary of the basics of MDWD elements and connections is given in Section 2; for a thorough treatment, we refer the reader to [1, 2, 5].

The resulting algorithms are similar to those of digital waveguide networks (DWNs) [6], in that they are composed of scattering junctions (called adaptors in the WDF context, but functionally identical) and shifting operations; all operations are performed on wave variables in either case. Indeed, DWNs can be directly applied to the same types of PDEs, namely those of hyperbolic type [7] (for which propagation speed is bounded). The two types of structures, are, however, not the same, though they both exhibit the same good numerical properties (a direct result of their underlying discrete passivity). In particular, DWNs can be shown to be equivalent to simple finite difference schemes [8], but MDWD networks can be identified with multi-step schemes of higher degree, and their respective numerical dispersion properties are distinct [9]. In addition, there has not as yet been a convenient MD representation for a DWN as there is for a MDWDF (i.e. a MD circuit), and thus no simple way of arriving at a DWN from a given system of PDEs.

It is shown in Section 3 that DWNs and MDWD networks can be unified in a straightforward way. In fact, a DWN may be derived from an MD circuit representation in a way nearly identical to the wave digital approach—the differences are in the form of the circuit representation and the type of spectral mapping used. We will examine, in particular, the (1+1)D transmission line system in the presence of distributed spatially-varying material parameters (inductance and capacitance). The same result may be applied in order to generate DWNs from the existing MD circuit representations for various more complex systems in higher dimensions. In particular, it is possible to arrive at DWNs which simulate various systems of interest in musical physical modelling applications, including beam [10, 9] plate and shell dynamics problems [9], as well as full (3+1)D elastic solid vibration [10, 9].

2. REVIEW OF MULTIDIMENSIONAL WAVE DIGITAL FILTERS

We briefly outline here the MDWD discretization procedure, as presented initially in [1, 2]; unfortunately, space limitations preclude a full review of lumped WDF principles (a prerequisite for the material in this section). We refer the reader to [3, 4] for a thorough introduction to WDFs.

An $(n+1)$D dynamic distributed problem is defined with respect to the variables $\mathbf{u} = [x_1, \ldots, x_n, t]^T$. Here, the x_j, $j = 1, \ldots, n$ are the n spatial coordinates, and t is physical time. (For simplicity, we will leave out any discussion of boundary conditions in this paper.) Generally speaking, the wave digital approach to numerical integration entails modelling a system of PDEs as a network of interconnected circuit elements which are themselves distributed objects. If the model system is passive, it is usually possible to represent it by a distributed network whose elements are all individually passive; sources (driving functions), if they are present, are excepted in this respect. Passivity-preserving spectral mappings or integration rules can then be applied to discretize the elements individually; when reconnected, these elements form a discrete passive network which simulates the model system.

2.1. Coordinate changes and sampling grids

In order to extend the notion of passivity to multiple dimensions (in which case it has been called *MD-passivity*), Fettweis and Nitsche introduced coordinate changes in [2]. The purpose of introducing these new coordinates is to ensure passivity with respect to all the coordinates in the problem; in the untransformed coordinates, passivity for most systems of interest holds with respect to time alone [2]. A useful type of coordinate change is given by

$$\mathbf{t} = \mathbf{H}^{-1}\mathbf{V}\mathbf{u} \tag{2}$$

for some new set of coordinates $\mathbf{t} = [t_1, \ldots, t_{n+1}]^T$, an orthogonal matrix $\mathbf{H}$, and a scaling matrix $\mathbf{V} = \mathrm{diag}(1, \ldots 1, v_0)$. A positivity constraint on the elements of the last row of $\mathbf{H}$ ensures that all of the new coordinates are all at least partially aligned with the positive time direction. v_0 later takes on an important role as the space-step/time-step ratio, and is subject to a stability condition, as per other explicit numerical methods [7].

Because we will be examining only a (1+1)D problem in this paper—that is, the transmission-line problem is defined with respect to coordinates $\mathbf{u} = [x, t]^T$—we introduce the transformed coordinates $\mathbf{t} = [t_1, t_2]^T$ defined by

$$\mathbf{H} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \tag{3}$$

Clearly, a positive change in either t_1 or t_2 implies a positive change in t, accompanied by a spatial shift. We also define the scaled time variable $t' = v_0 t$.

For a numerical scheme which is to be used to solve an $(n+1)$D problem, it will be necessary for it to operate on a finite subset of points in the domain. Such a grid can be obtained (in this context) by uniform sampling in the $\mathbf{t}$ coordinates. As an example, suppose that the (1+1)D coordinates defined above are uniformly sampled such that $(t_1, t_2) = (n_1 T_1, n_2 T_2)$ is a grid point for step-sizes T_1 and T_2, and for integer n_1 and n_2. Assuming that

$$T_1 = T_2 = \sqrt{2}\Delta \tag{4}$$

then in terms of the $\mathbf{u}$ coordinates, the grid points can be indexed by $(x, t) = (m\Delta, k\Delta/v_0)$, for integer m and k such that $m + k$ is even; here Δ is the grid-spacing in the untransformed coordinates, and Δ/v_0 is the time-step (which we can redefine as T). Information on grid generation in higher dimensions can be found in [2].

2.2. Circuit Elements

Wave digital filtering is essentially concerned with the discretization of connected *N-port* devices [11]; associated with the kth port, $k = 1, \ldots, N$ of such a device is a voltage v_k and a current i_k. In the distributed, or MD setting, we will in general have $v_k = v_k(\mathbf{t})$ and $i_k = i_k(\mathbf{t})$. In other words, the port quantities are themselves distributed.

The principal one-port MD circuit elements are the *inductor*, of inductance L and the *capacitor* of capacitance C; their definitions as well as schematic representations are as shown in Figure 1. Both are defined with respect to the directional derivative $D_j \triangleq \frac{\partial}{\partial t_j}$, where t_j, $j = 1, \ldots n+1$ is one of the transformed coordinate directions. t_j, in either case, indicates both the direction of *stored energy flow* [5] and, in the numerical context, an integration direction (see Subsection 2.5 for more information). In circuit models of systems of PDEs, such elements are usually linear and shift-invariant (LSI). It is usually possible to consolidate any spatial material parameter variation in an inductor or capacitor whose direction is t' (such an element is still linear and time-invariant, if not LSI). We have left out a discussion of the resistor [4], because we will work only with lossless systems in this paper.

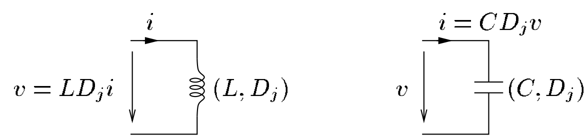

Figure 1: *Left: Inductor of inductance L, and direction t_j. Right: Capacitor of capacitance C and direction t_j.*

We also mention the standard lossless two-port elements, the ideal transformer and gyrator (see Figure 2), which are defined by

$$\text{Transformer}: \quad v_2 = n_T v_1 \qquad i_1 = -n_T i_2 \tag{5}$$

$$\text{Gyrator}: \quad v_1 = -R_G i_2 \qquad v_2 = R_G i_1 \tag{6}$$

Here, n_T is the transformer turns ratio, and R_G is the gyration coefficient, assumed positive. Both will play important roles in the circuit models of Section 3.

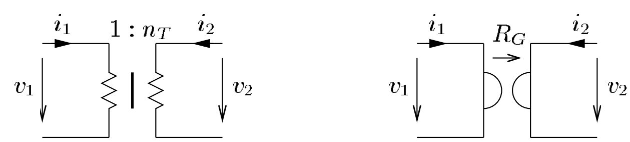

Figure 2: *Left: Transformer, of turns ration n_T. Right: Gyrator, of gyration constant R_G.*

2.3. Kirchoff's Laws

One requirement on the networks that we will discuss is that they be decomposable into portwise connections of N-ports, through Kirchoff's Laws. That is, for a series connection of any K ports, we require

$$v_1 + v_2 + \ldots + v_K = 0 \qquad i_1 = i_2 = \ldots = i_K \tag{7}$$

and for a parallel connection,

$$i_1 + i_2 + \ldots + i_K = 0 \qquad v_1 = v_2 = \ldots = v_K \tag{8}$$

A Kirchoff connection is a K-port in its own right.

2.4. Power and Passivity

The instantaneous power absorbed by an N-port with voltages v_k and currents i_k, $k = 1, \ldots, N$ is defined by

$$w_{inst} = \sum_{k=1}^{N} v_k i_k \tag{9}$$

This definition holds in MD, and in this case may be interpreted as an instantaneous absorbed power *density*. For the transformer and gyrator, w_{inst} is identically zero; these elements are thus *lossless*. The same is true of a Kirchoff connection. A full discussion of *MD-passivity* requires introducing the concept of stored energy flow, and a generalized definition of passivity. We refer the reader to [5] for details, and simply mention that the MD inductor and capacitor will be MD-passive (and lossless) for positive L or C, as long as L and C are shift-invariant with respect to the defining direction.

2.5. Discretization

Discretization, in the case of memoryless elements, such as the transformer, gyrator, or the Kirchoff connection, is straightforward. The element definitions remain the same, but the voltages and currents are assumed to take on values over a set of grid points (defined as in Subsection 2.1).

For reactive elements such as the inductor or capacitor, an approximation to the directional derivative is necessary. Fettweis et. al. have applied the trapezoid rule in a directional sense, i.e.,

$$D_j \rightarrow \frac{2}{T_j}(1+\delta_{\mathbf{T}_j})^{-1}(1-\delta_{\mathbf{T}_j}) \tag{10}$$

Here, $\delta_{\mathbf{T}_j}$ is a shift operator defined by

$$\delta_{\mathbf{T}_j}u(\mathbf{t}) = u(\mathbf{t}-\mathbf{T}_j) \tag{11}$$

when applied to a function $u(\mathbf{t})$. We have used $\mathbf{T}_j = T_j\mathbf{e}_j$, where $\mathbf{e}_j$ is a unit vector pointing in the direction t_j, and T_j is the *step-size*. As such, the equation for the inductor is discretized as

$$\frac{1}{2}\left(v(\mathbf{t})+v(\mathbf{t}-\mathbf{T}_j)\right) = \frac{L}{T_j}\left(i(\mathbf{t})-i(\mathbf{t}-\mathbf{T}_j)\right) \tag{12}$$

v and i may be approximated by grid functions. This rule approximates the equation of the continuous MD-inductor to second-order accuracy in T_j. The capacitor can be similarly discretized.

If a particular N-port to be discretized is LSI, then it is possible to interpret (12) in the frequency domain as a spectral mapping (at least, until it is connected to elements which are not LSI). For example, if s_j is the frequency variable conjugate to t_j, and z_j^{-1} is the transmittance of a unit shift in direction t_j, then the mapping can be written as

$$s_j \rightarrow \frac{2}{T_j}\frac{1-z_j^{-1}}{1+z_j^{-1}} \tag{13}$$

It is also necessary to use inductors and capacitors of direction t', in order to accommodate material parameter variation. In this case, it is permissible to apply rule (12) in a *generalized sense* [2], with D_j and $\delta_{\mathbf{T}_j}$ replaced by a time derivative $D_{t'}$ and a shift $\delta_{\mathbf{T}'}$ of length T' in direction t'. In this case L or C may have spatial dependence; passivity is still maintained.

2.6. Wave Variables

As in the lumped case, discretization via the trapezoid rule leads to discrete N-ports which may have a delay-free path from input to output, and can hence not be connected to others without the appearance of delay-free loops [3, 4]. For a given port, with voltage v and current i, it is possible to define *voltage wave variables* [4] by

$$a = v + iR \qquad \text{Input Wave} \tag{14a}$$
$$b = v - iR \qquad \text{Output Wave} \tag{14b}$$

The parameter $R>0$ is called the *port resistance*, and is to be defined separately for each port; it is introduced as an extra degree of freedom which can be used to remove delay-free paths, if wave variables are employed as the signals in the network, instead of voltages and currents. The resulting N-port is called a wave-digital N-port. For the MD inductor of inductance L, and direction t_j, it is straightforward to show that for a choice of $R = 2L/T_j$, the wave digital one-port of Figure 3 results. For the MDWD inductor, a signal a enters, is shifted by $\mathbf{T_j}$, sign-inverted, and exits as b. It is important to keep in mind that the input and output waves of any MDWD one-port are grid functions; thus in the case of the inductor, it is an entire array of signals which is shifted and sign-inverted. The wave-digital capacitor is also shown in Figure 3.

Figure 3: *Left: MDWD inductor, of direction t_j and port resistance $R = 2L/T_j$. Right: MDWD capacitor, of direction t_j and port resistance $R = T_j/2C$.*

Kirchoff's Laws for a connection of K ports can be written in terms of wave variables as

$$b_k = a_k - \frac{2R_k}{\sum_{m=1}^{K}R_m}\sum_{p=1}^{K}a_p \qquad k=1,\ldots,K \qquad \text{(Series)}$$

$$b_k = -a_k + \frac{2}{\sum_{m=1}^{K}G_m}\sum_{p=1}^{K}G_p a_p \qquad k=1,\ldots,K \qquad \text{(Parallel)}$$

where $G_p = 1/R_p$ for any port p. This scattering operation (or signal-processing block) is represented, as in the lumped case, as per Figure 4.

Figure 4: *Left: Series K-port adaptor (scattering junction). Right: Parallel K-port adaptor.*

2.7. Multidimensional Unit Elements

In order to bridge the gap between MDWDFs and digital waveguide networks, it will be necessary to define the multidimensional unit element. Referring to (1+1)D coordinates defined by (3), such an element is shown in Figure 5. This MDWD two-port is clearly lossless (indeed, its action is merely to shift the signal arrays entering from the left- and right-hand ports by $\mathbf{T}_1$ and $\mathbf{T}_2$ respectively), and its port resistance are both assumed equal to some positive constant R. As such, it is an MD representation of an array of bidirectional delay lines [6] of impedance R. We note that we have generalized here the definition of multidimensional unit elements proposed in [12]. Because this two-port is LSI, we may analyze it

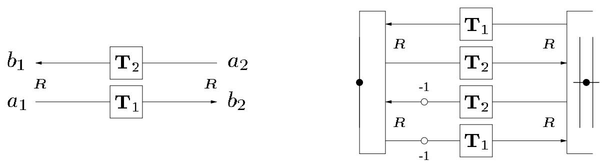

Figure 5: *Left: MD unit element, with shifts* $\mathbf{T}_1$ *and* $\mathbf{T}_2$. *Right: Series/parallel connection of two MD unit elements, one with sign inversion.*

in the frequency domain; it is defined by

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix} = \begin{bmatrix}0 & z_2^{-1}\\ z_1^{-1} & 0\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix} \tag{15}$$

in terms of the steady-state wave quantities (hatted). We may invert the definition of wave variables (14), in order to obtain a discrete impedance relation,

$$\begin{bmatrix}\hat{v}_1^d\\ \hat{v}_2^d\end{bmatrix} = \frac{R}{1-z_1^{-1}z_2^{-1}}\begin{bmatrix}1+z_1^{-1}z_2^{-1} & 2z_2^{-1}\\ 2z_1^{-1} & 1+z_1^{-1}z_2^{-1}\end{bmatrix}\begin{bmatrix}\hat{i}_1^d\\ \hat{i}_2^d\end{bmatrix} \tag{16}$$

in terms of the discrete steady-state voltages $\hat{v}_1^d$ and $\hat{v}_2^d$ and currents $\hat{i}_1^d$ and $\hat{i}_2^d$. Note that the MD unit element is not reciprocal [11]. It is useful to rewrite (16) in a *hybrid form*, as

$$\mathbf{p} = \mathbf{H}_{ue}(z_1^{-1}, z_2^{-1})\mathbf{q} \tag{17}$$

where $\mathbf{p} = [\hat{v}_1^d, \hat{i}_2^d]^T$ and $\mathbf{q} = [\hat{i}_1^d, \hat{v}_2^d]^T$ and

$$\mathbf{H}_{ue}(z_1^{-1}, z_2^{-1}) = \begin{bmatrix}\frac{R\left(1-z_1^{-1}z_2^{-1}\right)}{1+z_1^{-1}z_2^{-1}} & \frac{2z_2^{-1}}{1+z_1^{-1}z_2^{-1}}\\ \frac{-2z_1^{-1}}{1+z_1^{-1}z_2^{-1}} & \frac{1+z_1^{-1}z_2^{-1}}{R\left(1+z_1^{-1}z_2^{-1}\right)}\end{bmatrix} \tag{18}$$

where the order of the arguments of $\mathbf{H}_{ue}$ is enforced. As a prelude to the material in Subsection 3.2, we also examine a pair of MD unit elements whose left-hand ports are connected in series, and whose right-hand ports are connected in parallel (see Figure 5). The two MD unit elements have opposite orientations, and one incorporates a pair of sign inversions. These two unit elements may be considered together as a single two-port; its hybrid matrix relation [11] is

$$\mathbf{p} = \left(\mathbf{H}_{ue}(z_1^{-1}, z_2^{-1}) + \mathbf{H}_{ue}(-z_2^{-1}, -z_1^{-1})\right)\mathbf{q} \tag{19}$$

(for a series/parallel combination of two two-ports, hybrid matrices sum, as may be easily verified [9]).

2.8. Alternative Spectral Mappings

One might well ask whether the bilinear transform or trapezoid rule is the only way of passively discretizing a given passive N-port. In fact, other such mappings or integration rules are available. In the case of the (1+1)D coordinates defined by (3), consider the pair of spectral mappings defined by

$$s_1 \rightarrow \frac{1}{T_1}\frac{(1-z_1^{-1})(1+z_2^{-1})}{1+z_1^{-1}z_2^{-1}} \tag{20a}$$

$$s_2 \rightarrow \frac{1}{T_2}\frac{(1-z_2^{-1})(1+z_1^{-1})}{1+z_1^{-1}z_2^{-1}} \tag{20b}$$

Mappings similar to these were mentioned very briefly in [2]. Though it is straightforward to show the passivity-preserving property of these mappings, we forego a formal treatment here and merely note that passivity is obvious from the structures that result from their application. They are also second-order accurate in the spacings T_1 and T_2.

3. THE (1+1)D TRANSMISSION LINE EQUATIONS

The lossless, source-free (1+1)D transmission line equations, defined by

$$l\frac{\partial i}{\partial t} + \frac{\partial u}{\partial x} = 0 \tag{21a}$$

$$c\frac{\partial u}{\partial t} + \frac{\partial i}{\partial x} = 0 \tag{21b}$$

provide a useful model problem for the scattering methods to be discussed in this paper [13]. Here, $l(x) > 0$ and $c(x) > 0$ are the inductance and capacitance per unit length and $u(x,t)$ and $i(x,t)$ are the voltage across and current in the line. We neglect boundary conditions here, so that the problem is assumed to be defined for $-\infty < x < \infty$. This problem can be simplified to include (1+1)D systems of interest in musical sound synthesis, including acoustic tubes, and lossless strings (with possibly spatially-varying density) etc. Loss and source terms can be easily added to system (21). We remark that system (21) requires two initial conditions; initialization of scattering methods in the numerical context is discussed in detail in [9].

3.1. Multidimensional Kirchoff Circuit and MDWD network

The MDKC for the transmission line equations is shown at the top left of Figure 6. It is important to realize that this circuit is simply a graphical representation of system (21). It is itself a multidimensional object; in particular it is not a circuit which can be "built." The currents in the two loops are defined by

$$i_1 = i \qquad\qquad i_2 = u/r_0 \tag{22}$$

where r_0 is a positive constant which has dimensions of resistance, and which is used to "tune" the resulting MDWD structure so as to achieve maximum computational efficiency [2]. Kirchoff's Voltage Law applied around the two loops yields system (21)

When approaching the discretization of the MDKC at the top of Figure 1, it is best to think of it as a two-port $PP'QQ'$ terminated on two one-port inductors, of inductances L_1 and L_2. Each of these three N-ports is discretized separately, and then all are reconnected to one-another via adaptors, or scattering junctions. The two one-port inductors in direction t' become delays (with sign-inversion). The so-called "Jaumann" two-port $PP'QQ'$ [2] deserves a bit more commentary. Referring to Figure 6, it is defined by the relation

$$\begin{bmatrix}v_1\\ v_2\end{bmatrix} = L_0\begin{bmatrix}D_1+D_2 & D_1-D_2\\ D_1-D_2 & D_1+D_2\end{bmatrix}\begin{bmatrix}i_1\\ i_2\end{bmatrix} \tag{23}$$

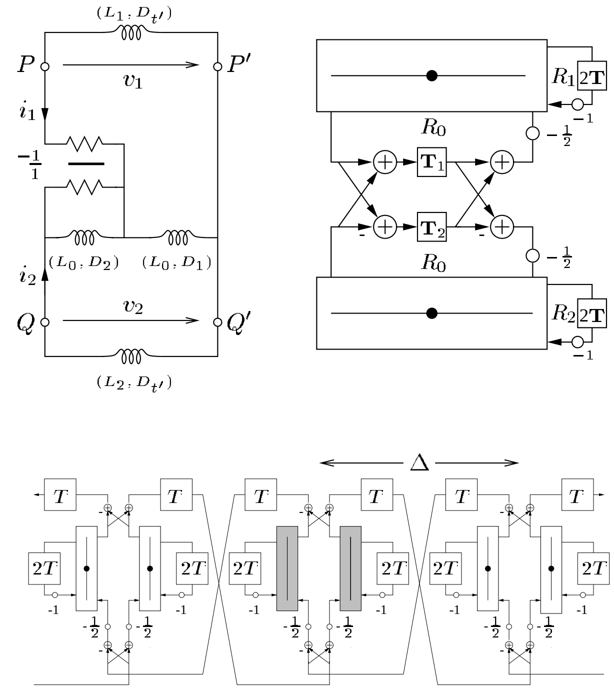

Figure 6: *Top left: MDKC for the (1+1)D transmission line system* (21)*, with $L_1 = v_0 l - r_0$, $L_2 = v_0 r_0^2 c - r_0$, and $L_0 = r_0/\sqrt{2}$. Top right: MDWD network, with $R_1 = 2L_1/T'$, $R_2 = 2L_2/T'$ and $R_0 = 4L_0/T_1$, where we have used the step sizes of* (4)*, as well as $T' = 2\Delta$ (see comment at end of Section 2.5). Bottom: Expanded signal-flow diagram (port-resistances not shown). Grey/white coloration of junctions indicates that calculations are to be performed at alternating time instants.*

Because this two-port is LSI (L_0 is a constant), it is permissible to work with steady-state amplitudes $\hat{v}_1$, $\hat{v}_2$, $\hat{i}_1$ and $\hat{i}_2$, and the frequency variables s_1 and s_2, in which case (23) becomes

$$\begin{bmatrix}\hat{v}_1\\ \hat{v}_2\end{bmatrix} = L_0 \begin{bmatrix} s_1+s_2 & s_1-s_2 \\ s_1-s_2 & s_1+s_2\end{bmatrix}\begin{bmatrix}\hat{i}_1\\ \hat{i}_2\end{bmatrix} \tag{24}$$

Employing wave variables defined by (14), with port resistances $R_0 = 4L_0/T_1$, and the spectral mappings as defined by (13) yields the scattering relation

$$\begin{bmatrix}\hat{b}_1\\ \hat{b}_2\end{bmatrix} = -\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix} z_1^{-1} & 0\\ 0 & z_2^{-1}\end{bmatrix}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}\hat{a}_1\\ \hat{a}_2\end{bmatrix} \tag{25}$$

The corresponding signal-flow graph for this discrete two-port which connects the two series adaptors is as shown in Figure 6. The resulting structure is again to be interpreted in a multidimensional sense, and is a short-hand notation for an integration scheme. In a computer implementation, at a given grid location, the directional shifts $\mathbf{T}_1$ and $\mathbf{T}_2$ refer to neighboring grid points. When the spatial dependence of this diagram is expanded out, the full signal flow graph appears as in the bottom of Figure 1 (we have chosen the step-sizes such that an *offset sampled* [1] algorithm results). The lower bound on v_0 which is sufficient for stability follows immediately from the positivity of the inductances L_1 and L_2. For the special choice of

$$r_0 = \sqrt{\frac{\min_x l}{\min_x c}} \tag{26}$$

the bound is

$$v_0 \geq \sqrt{\frac{1}{\min_x l \min_x c}} \tag{27}$$

3.2. An Alternative MDKC and Digital Waveguide Network

We now reexamine the MDKC of Figure 6, and note that the lower one-port inductor, with terminals Q and Q' is equivalent to a gyrator of gyration coefficient r_0 terminated on a capacitor; the transformed MDKC is shown in Figure 7, and we note that although its topology is different from that of Figure 6, it also represents system (21). Also, we have scaled the system (21) by a factor of 2Δ (the grid-spacing); this has no influence on the numerical solution, and merely makes the transition to DWNs simpler. We may consider this network to be a two-port $XX'YY'$, terminated in series on an inductor, now of inductance $L_1 = 2\Delta\,(v_0 l - r_0)$, and in parallel on a capacitor of capacitance $C_2 = 2\Delta\,(v_0 c - 1/r_0)$ Discretization of the one-port inductor and capacitor may be performed using the trapezoid rule as before, with a step size of $T' = 2\Delta$; the LSI connecting two-port $XX'YY'$, however, is treated somewhat differently.

The continuous hybrid relation for this two-port is

$$\begin{bmatrix}\hat{v}_1\\ \hat{i}_2\end{bmatrix} = \sqrt{2}\Delta\begin{bmatrix} r_0\,(s_1+s_2) & s_1-s_2\\ s_1-s_2 & \frac{1}{r_0}\,(s_1+s_2)\end{bmatrix}\begin{bmatrix}\hat{i}_1\\ \hat{v}_2\end{bmatrix} \tag{28}$$

Under the application of the spectral mappings (20), with step sizes again given by (4), the discrete hybrid relation is exactly (19), with $R = r_0$. Thus the discrete image of the two-port $XX'YY'$ decomposes into a series/parallel combination of MD unit elements; the discrete MD network, as well as the expanded flow graph are shown in Figure 7. At this point, we mention that the port resistances r_0 can, in the DWN context, be interpreted as *waveguide impedances*. The stability bound on v_0 is unchanged from (27), if r_0 is chosen as per (26). Also, if l and c are constants, and (27) holds with equality, then L_1 and C_2 are zero, so the one-port inductance and capacitance can be dropped from the network entirely, leaving (as expected) a signal flow-graph of the form of a single bidirectional delay line (the standard form of the DWN which implements a traveling-wave solution to the (1+1)D wave equation).

4. CONCLUSIONS

There are three main conclusions that can be drawn here:

1) Though we have examined only a simple example in this paper, it can be shown that the complete theory underlying the construction of passive wave digital networks for the simulation of physical systems can be carried over directly to digital waveguide networks [9]. This is important, because this theory is well-developed, and makes use of the powerful concept of MD-passivity [5]. We note, however, that there are DWNs, in particular those which operate on irregular grids, which can not be derived from an MDKC [9]. Also, there exist forms of the DWN for which the waveguide impedances may vary spatially [8, 9].

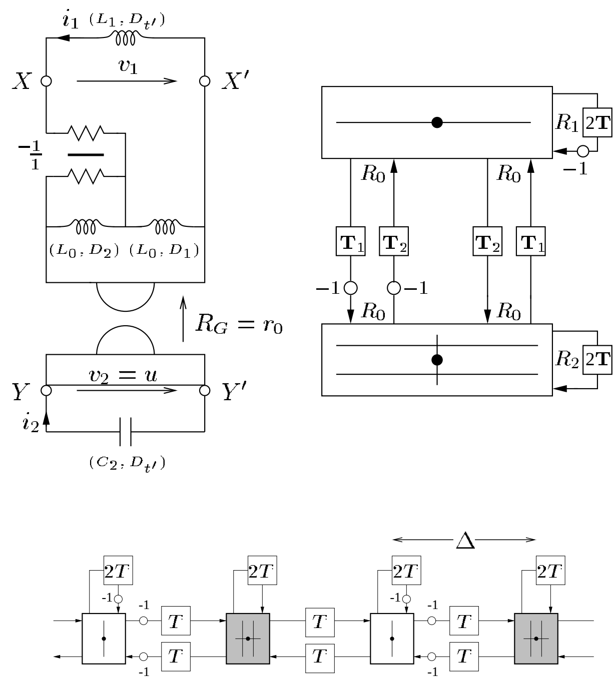

Figure 7: *Top left: Transformed MDKC for the (1+1)D transmission line system* (21), *with* $L_1 = 2\Delta(v_0 l - r_0)$, $C_2 = 2\Delta(v_0 c - 1/r_0)$, *and* $L_0 = \sqrt{2}\Delta r_0$. *Top right: MDWD network, with* $R_1 = 2L_1/T'$, $R_2 = T'/(2C_2)$ *and* $R_0 = r_0$. *Bottom: Expanded signal-flow diagram (port-resistances not shown). Grey/white coloration of junctions indicates calculation at alternating time instants.*

2) A wide range of physical systems have been successfully modeled using circuit representations. Besides the systems mentioned in this paper, it is possible to develop MDWD simulation schemes for beam [10, 9], plate and shell dynamics systems [9], Maxwell's Equations [10], and even nonlinear fluid dynamics systems [14]. DWNs are now also feasible alternative integration methods for all these systems, and possess all the good numerical properties of MDWDFs.

3) Most importantly, it should be clear that by applying transformations from classical network theory to an MDKC, and through the use of passivity-preserving mappings (regardless of their form), passivity of the resulting simulation routine is never compromised. As such, the door is opened to a wide range of schemes which will all be composed of the same robust signal processing blocks (in particular, they all will make use of scattering junctions), and will propagate wave variables; these forms may be quite different from MDWD or DWN networks.

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